The Bell–Szekeres Solution and Related Solutions of the Einstein–Maxwell Equations

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Abstract

A novel technique for solving some head–on collisions of plane homogeneous light–like signals in Einstein–Maxwell theory is described. The technique is a by–product of a re–examination of the fundamental Bell–Szekeres solution in this field of study. Extensions of the Bell–Szekeres collision problem to include light–like shells and gravitational waves are described and a family of solutions having geometrical and topological properties in common with the Bell–Szekeres solution is derived.

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1 Introduction

The simplest collision space–time known involving homogeneous plane electromagnetic waves in Einstein–Maxwell theory is arguably the head–on collision of electromagnetic shock waves having a step function profile given by Bell and Szekeres [1]. The Bell–Szekeres solutions satisfy the vacuum Einstein–Maxwell field equations while the resulting space–times admit a pair of space–like, hypersurface–orthogonal, commuting Killing vector fields. The line–element of the region of the Bell–Szekeres space–time following the collision of the shock waves turned out to be identical to the line–element found independently by Bertotti [2] and by Robinson [3], whose motivation was purely geometrical. Global properties of the Bell–Szekeres solutions were subsequently described by Clarke and Hayward [4].

The physical interpretation of the Bell–Szekeres solutions is that they describe the electromagnetic and gravitational fields before and after the head–on collision of two linearly polarized, homogeneous, plane fronted electromagnetic shock waves which (a) have a step function profile and (b) which do not interact before collision.

By analogy with the Weyl static, axially symmetric space–times or more generally the stationary, axially symmetric space–times, which admit two commuting Killing vector fields (one space–like and one time–like) which can be described in the formulation of Ernst [5] [6], the space–times resulting from the head–on collision of plane fronted, homogeneous, light–like signals can also be described in an Ernst formalism (see Chandrasekhar [7], Chandrasekhar and Ferrari [8] and Chandrasekhar and Xanthopoulos [9]).

In this paper we consider the problem of finding the space–time after the head–on collision of two homogeneous, plane fronted, light–like signals, each of which incorporates an electromagnetic shock wave of the Bell–Szekeres type, an impulsive gravitational wave and a light–like shell of matter. The solution to this problem is in general unknown and, among other special cases, it must include the Bell–Szekeres solutions. We present in this paper new solutions of the Einstein–Maxwell field equations which constitute a subclass of this general problem for which there exist algebraic relations between the parameters describing the different physical components of the incoming light–like signals. Our family of solutions contains the Bell–Szekeres solutions as a special case and from a geometrical and topological point of view are shown to be closely related to the Bell–Szekeres space–times. We make use of a technique for solving the field equations which we were led to while re–constructing the Bell–Szekeres space–times. No derivation of the solutions is described in the original paper by Bell and Szekeres. Our technique is both simple and new and has the possibility of being used to find further new solutions of these field equations.

The outline of the paper is as follows: In section 2 we describe a novel technique for finding collision solutions of the Maxwell and Einstein–Maxwell vacuum field equations and illustrate it by giving a derivation of the Bell–Szekeres solution. In section 3 we state the general collision problem in which the electromagnetic shock waves of Bell–Szekeres are accompanied by light–like shells of matter and impulsive gravitational waves. In commenting on the general problem we describe a new collision solution which solves a subclass of the general problem. A subclass of the general collision problem described in section 3 which includes the Bell–Szekeres solution as a special case is derived in detail in section 4 making use of the technique given in section 2. Consequences of conformal flatness are worked out in the present context in section 5 and establish the uniqueness of the collision problem solved in section 4. A topological property of the family of solutions obtained in section 4 is discussed in section 6.

2 The Bell–Szekeres Solution Revisited

As in all collision problems involving the head–on collision of homogeneous plane light–like signals, going back to the pioneering work of Szekeres [10][11], Khan and Penrose [12] and Bell and Szekeres [1] we start with the Rosen–Szekeres form of line–element:

$$ds^2 = -e^{-U}\,(e^{V}\,dx^2 + e^{-V}\,dy^2) + 2\,e^{-M}\,du\,dv\ , \tag{2.1}$$

where the functions U, V, M depend on the coordinates u, v only. The Maxwell field is described in Newman–Penrose notation by two real–valued functions ϕ_0 and ϕ_2 (depending only on u, v) satisfying Maxwell's vacuum field equations,

$$\begin{aligned}\frac{\partial\phi_2}{\partial v} &= \frac{1}{2}U_v\,\phi_2 - \frac{1}{2}V_u\,\phi_0\ , &(2.2)\\ \frac{\partial\phi_0}{\partial u} &= \frac{1}{2}U_u\,\phi_0 - \frac{1}{2}V_v\,\phi_2\ . &(2.3)\end{aligned}$$

Here and throughout subscripts denote partial derivatives. The functions appearing in the line–element (2.1) satisfy the Einstein–Maxwell vacuum field equations:

$$\begin{aligned}U_{uv} &= U_u\,U_v\ , &(2.4)\\ 2\,U_{uu} &= U_u^2 + V_u^2 - 2\,U_u\,M_u + 4\,\phi_2^2\ , &(2.5)\\ 2\,U_{vv} &= U_v^2 + V_v^2 - 2\,U_v\,M_v + 4\,\phi_0^2\ , &(2.6)\end{aligned}$$

$$
\begin{aligned}
2\,V_{uv} &= U_u\,V_v + U_v\,V_u + 4\,\phi_0\,\phi_2\ , && (2.7)\\
2\,M_{uv} &= V_u\,V_v - U_u\,U_v\ . && (2.8)
\end{aligned}
$$

It is well–known that the first of these is solved in general by

$$
e^{-U} = f(u) + g(v)\ , \tag{2.9}
$$

and the functions f and g are immediately determined by the initial (boundary) conditions. As a final preliminary we note that the Newman–Penrose components of the Weyl conformal curvature tensor calculated with the metric given via the line–element (2.1) are

$$
\begin{aligned}
\Psi_0 &= -\frac{1}{2}\,(V_{vv} - U_v\,V_v + M_v\,V_v)\ , && (2.10)\\
\Psi_1 &= 0\ , && (2.11)\\
\Psi_2 &= \frac{1}{4}\,(V_u\,V_v - U_u\,U_v)\ , && (2.12)\\
\Psi_3 &= 0\ , && (2.13)\\
\Psi_4 &= -\frac{1}{2}\,(V_{uu} - U_u\,V_u + M_u\,V_u)\ . && (2.14)
\end{aligned}
$$

We describe here a simple technique to solve the field equations (2.2)–(2.8) in special circumstances. Since the technique arose in our re–examination of the Bell–Szekeres solution we will illustrate it by using it to give a derivation of the Bell–Szekeres solution. We begin by re–writing (2.2) in the form

$$
\frac{\partial}{\partial v}(\log\phi_2) = \frac{1}{2}U_v - \frac{1}{2}V_u\,\frac{\phi_0}{\phi_2}\ , \tag{2.15}
$$

and re–writing (2.3) in the form

$$
\frac{\partial}{\partial u}(\log\phi_0) = \frac{1}{2}U_u - \frac{1}{2}V_v\,\frac{\phi_2}{\phi_0}\ . \tag{2.16}
$$

From these we deduce that

$$
2\,\frac{\partial^2}{\partial u\partial v}(\log\frac{\phi_2}{\phi_0}) = \frac{\partial}{\partial v}\left(V_v\,\frac{\phi_2}{\phi_0}\right) - \frac{\partial}{\partial u}\left(V_u\,\frac{\phi_0}{\phi_2}\right)\ . \tag{2.17}
$$

We note that all of the equations given so far are invariant under the transformations $u \to \bar{u} = \bar{u}(u)$ and $v \to \bar{v} = \bar{v}(v)$. Under these transformations the functions ϕ_0, ϕ_2 and M transform as

$$
\phi_0 \to \bar{\phi}_0\ , \qquad \phi_2 \to \bar{\phi}_2\ , \qquad M \to \bar{M}\ , \tag{2.18}
$$

with

$$\phi_0 = \frac{d\bar{v}}{dv}\,\bar{\phi}_0\ , \qquad \phi_2 = \frac{d\bar{u}}{du}\,\bar{\phi}_2\ , \qquad e^{\bar{M}} = e^{M}\,\frac{d\bar{u}}{du}\,\frac{d\bar{v}}{dv}\ . \tag{2.19}$$

We will be interested in seeking solutions of (2.17) for which

$$\frac{\phi_2}{\phi_0} = \frac{A(u)}{B(v)}\ , \tag{2.20}$$

for some functions $A(u)$ and $B(v)$. In this case we can use (2.18) and (2.19) to choose a frame $(\bar{u}, \bar{v})$ for which

$$\bar{\phi}_0 = \bar{\phi}_2\ . \tag{2.21}$$

After the head–on collision of the electromagnetic waves the functions ϕ_0 and ϕ_2 describe back–scattered electromagnetic waves and (2.21) implies that there exists a frame of reference in which the energy densities of these back–scattered waves are equal. In this frame the equation (2.17) becomes a wave equation for V:

$$V_{\bar{u}\bar{u}} = V_{\bar{v}\bar{v}}\ . \tag{2.22}$$

If when $\bar{v} = 0$ we have the initial data, $V = P(\bar{u})$, $V_{\bar{v}} = Q(\bar{u})$ then V is given for $\bar{u} > 0$, $\bar{v} > 0$ by the d'Alembert formula:

$$V(\bar{u}, \bar{v}) = \frac{1}{2}\left\{P(\bar{u}+\bar{v}) + P(\bar{u}-\bar{v})\right\} + \frac{1}{2}\int_{\bar{u}-\bar{v}}^{\bar{u}+\bar{v}} Q(\xi)\,d\xi\ . \tag{2.23}$$

To illustrate the method in the previous paragraph we take the Bell–Szekeres problem. This consists of looking for the space–time following the head–on collision of two electromagnetic shock waves each having a step function profile. In terms of the coordinates introduced at the beginning of this section $\{x, y, u, v\}$ and the functions U, V, M, ϕ_0, ϕ_2 this problem is expressed mathematically by requiring a solution of the Einstein–Maxwell field equations (2.2)–(2.8) with the initial conditions: for $v = 0,\ u > 0$ we require

$$e^{U} = 1 + a^2u^2\ ,\ V = 0\ ,\ e^{M} = 1 + a^2u^2\ ,\ \phi_2 = \frac{a}{1 + a^2u^2}\ , \tag{2.24}$$

where a is a real constant, and for $u = 0,\ v > 0$ we require

$$e^{U} = 1 + b^2v^2\ ,\ V = 0\ ,\ e^{M} = 1 + b^2v^2\ ,\ \phi_0 = \frac{b}{1 + b^2v^2}\ , \tag{2.25}$$

where b is a real constant. Now by (2.9) we have trivially

$$e^{-U} = (1 + a^2u^2)^{-1} + (1 + b^2v^2)^{-1} - 1 = \frac{1 - a^2b^2u^2v^2}{(1 + a^2u^2)\,(1 + b^2v^2)}\ . \tag{2.26}$$

Also evaluating (2.2), (2.3) and (2.7) on $v=0$ and on $u=0$ we can obtain the following: on $v=0,\ u>0$ we find that

$$V_v = 2\,a\,b\,u\ , \qquad \phi_0 = b\ , \tag{2.27}$$

while on $u=0,\ v>0$ we have

$$V_u = 2\,a\,b\,v\ , \qquad \phi_2 = a\ , \tag{2.28}$$

where, as always a subscript denotes a partial derivative. Using this data the assumption (2.20) yields

$$\frac{\phi_2}{\phi_0} = \frac{a\,(1+b^2v^2)}{b\,(1+a^2u^2)}\ , \tag{2.29}$$

from which we deduce (2.21) with

$$b\,v = \tan\bar{v}\ , \qquad a\,u = \tan\bar{u}\ . \tag{2.30}$$

Now we apply the d'Alembert formula (2.23) with $P(\bar{u})=0$ and $Q(\bar{u}) = 2\,\tan\bar{u}$ to obtain

$$V = \log\left(\frac{\cos(\bar{v}-\bar{u})}{\cos(\bar{u}+\bar{v})}\right) = \log\left(\frac{1+a\,b\,u\,v}{1-a\,b\,u\,v}\right)\ . \tag{2.31}$$

Now using (2.26), (2.29) and (2.31) we easily see that

$$U_v\,\phi_2 - V_u\,\phi_0 \ = \ 0\ , \tag{2.32}$$

$$U_v\,\phi_0 - V_v\,\phi_2 \ = \ 0\ . \tag{2.33}$$

Hence Maxwell's equations (2.2) and (2.3) reduce to

$$\frac{\partial\phi_2}{\partial v} = 0\ ,\ \text{and}\ \ \frac{\partial\phi_0}{\partial u} = 0\ , \tag{2.34}$$

and so using the initial conditions (2.24) and (2.25) we have, for $u>0\ ,\ v>0$,

$$\phi_2 = \frac{a}{1+a^2u^2}\ , \qquad \phi_0 = \frac{b}{1+b^2v^2}\ . \tag{2.35}$$

We now substitute the functions $U,\ V,\ \phi_2\ ,\ \phi_0$ given by (2.26), (2.31) and (2.35) respectively into the remaining field equations (2.5)–(2.8). Equations (2.5) and (2.6) reduce to

$$M_u \ = \ \frac{2\,b^2v}{1+b^2v^2}\ , \tag{2.36}$$

$$M_v \ = \ \frac{2\,a^2u}{1+a^2u^2}\ . \tag{2.37}$$

Equation (2.7) is identically satisfied while equation (2.8) becomes

$$M_{uv} = 0 \ , \tag{2.38}$$

which is clearly consistent with (2.36) and (2.37). With the boundary conditions (2.24) and (2.25) we have, for $u > 0\ ,\ v > 0$,

$$M = \log(1 + a^2u^2)\,(1 + b^2v^2) \ . \tag{2.39}$$

The functions (2.26), (2.31), (2.35) and (2.39) constitute the Bell–Szekeres solution of the Einstein–Maxwell vacuum field equations. Calculation of the Weyl tensor components (2.10)–(2.14) reveals that they all vanish and so the Bell–Szekeres space–time $u > 0\ ,\ v > 0$ after the collision of the electromagnetic shock waves is conformally flat.

3 Some Einstein–Maxwell Space–Times

In the Bell–Szekeres example the histories of the wave fronts of the two families of incoming electromagnetic shock waves have equations $u = \text{constant} \geq 0\ ,\ v < 0$ and $v = \text{constant} \geq 0\ ,\ u < 0$. It is a simple matter to add to these signals light–like shells and impulsive gravitational waves with histories $u = 0\ ,\ v < 0$ and $v = 0\ ,\ u < 0$. This is done by modifying the initial conditions (2.24) and (2.25) to read (see Appendix A):for $v = 0\ ,\ u > 0$ we require

$$\begin{aligned} e^{-U} &= \frac{1 - 2\,l\,u + (l^2 - k^2)\,u^2}{1 + a^2u^2} \ , && (3.1)\\ e^{V} &= \frac{1 + (k - l)\,u}{1 - (k + l)\,u} \ , && (3.2)\\ e^{M} &= 1 + a^2u^2 \ , && (3.3)\\ \phi_2 &= \frac{a}{1 + a^2u^2} \ , && (3.4) \end{aligned}$$

and for $u = 0\ ,\ v > 0$ we require

$$\begin{aligned} e^{-U} &= \frac{1 - 2\,p\,v + (p^2 - s^2)\,v^2}{1 + b^2v^2} \ , && (3.5)\\ e^{V} &= \frac{1 + (s - p)\,v}{1 - (s + p)\,v} \ , && (3.6)\\ e^{M} &= 1 + a^2u^2 \ , && (3.7)\\ \phi_0 &= \frac{b}{1 + b^2v^2} \ . && (3.8) \end{aligned}$$

The parameters here have the following physical associations (in the sense that if any of the parameters are put equal to zero then that part of the light–like signal is removed): The parameters a, b label the incoming electromagnetic shock waves as in section 2. The parameters l, p label incoming light–like shells while the parameters k, s label incoming impulsive gravitational waves. *This collision problem with all parameters non–zero is unsolved.* Many special cases are of course solved, most fundamentally the Bell–Szekeres [1] ($l = p = k = s = 0$) case and the Khan–Penrose [12] ($a = b = l = p = 0$) case. Few cases in which at least one electromagnetic shock wave is present have been solved. The solution for a collision involving one electromagnetic shock wave and two impulsive gravitational waves was derived by Barrabès, Bressange and Hogan [13] (see also [14]). We have also found the solution for a collision involving one electromagnetic shock wave labelled by b and two light–like shells labelled by l and p (thus corresponding to $a = k = s = 0$ above). It is given, for $u > 0$, $v > 0$ by

$$e^{-U} = (1 - l\,u)^2 + \frac{(1 - p\,v)^2}{1 + b^2v^2} - 1\ , \tag{3.9}$$

$$e^{-M-\frac{1}{2}U} = \frac{(1 - p\,v)\,(1 - l\,u)}{(1 + b^2v^2)^{3/2}}\ , \tag{3.10}$$

$$\phi_0\, e^{-\frac{1}{2}U} = \frac{b\,(1 - p\,v)}{(1 + b^2v^2)^{3/2}}\ , \tag{3.11}$$

and, in addition $V = 0 = \phi_2$.

Our objective in this paper is to solve the Einstein–Maxwell initial value problem with initial data (3.1)–(3.8) and obtain solutions which include the Bell–Szekeres solution as a special case. Our strategy to achieve this is to focus on some feature of the Bell–Szekeres space–time which we can impose on the collision space–time that we are looking for. Perhaps the simplest feature of the Bell–Szekeres space–time is its conformal flatness so we will look for conformally flat solutions of the Einstein–Maxwell vacuum field equations with the initial data (3.1)–(3.8). We implement this requirement in a gradual way which we describe in detail in the next section.

4 Generalizations of the Bell–Szekeres Solution

We begin by searching for necessary conditions for conformal flatness of the collision space–time. The simplest is obtained by requiring Ψ_2 in (2.12) to vanish at $u = 0 = v$ (strictly speaking in the limits $u \to 0^+$ and $v \to$

0^+). With the initial data (3.1)–(3.8) this requirement yields the following relationship between the parameters:

$$p\,l = s\,k\ . \tag{4.1}$$

Another piece of useful information can be obtained by requiring Ψ_2 to vanish *near* $v = 0 = u$. To find this we we need to calculate ϕ_0 and V_v when $v = 0$. The differential equations for these quantities are obtained by evaluating (2.3) and (2.7) at $v = 0$ and using the initial data. We find that at $v = 0\ , u > 0$:

$$\phi_0 = \frac{p\,a}{k} + \frac{(b\,k - a\,p)}{k\,\sqrt{1 - 2\,l\,u + (l^2 - k^2)\,u^2}}\ , \tag{4.2}$$

and

$$V_v = \frac{2\,(b\,k - a\,p)\,a\,u}{k\,\sqrt{1 - 2\,l\,u + (l^2 - k^2)\,u^2}} + \frac{2\,p\,\{l - (l^2 - k^2 - a^2)\,u + l\,a^2u^2\}}{k\,(1 - 2\,l\,u + (l^2 - k^2)\,u^2)}\ . \tag{4.3}$$

At this point, on account of the well–known fact (2.9), the function U is given for $u > 0\ ,\ v > 0$, by

$$e^{-U} = \frac{1 - 2\,l\,u + (l^2 - k^2)\,u^2}{1 + a^2u^2} + \frac{1 - 2\,p\,v + (p^2 - s^2)\,v^2}{1 + b^2v^2} - 1\ . \tag{4.4}$$

Using these we find that at $v = 0$

$$\Psi_2 = \frac{(b\,k - a\,p)\,a\,u}{(1 - 2\,l\,u + (l^2 - k^2)\,u^2)^{3/2}}\ , \tag{4.5}$$

from which we conclude that, in addition to (4.1) a further necessary condition for conformal flatness is

$$b\,k = a\,p\ . \tag{4.6}$$

Eq.(4.1) is symmetrical under interchange of the light–like shell (l) and impulsive gravitational wave (k) accompanying the electromagnetic shock wave (a) with the light–like shell (p) and impulsive gravitational wave (s) accompanying the electromagnetic shock wave (b) [assuming these shells and gravitational waves exist]. This suggests that (4.6) should have a partner equation which is obtained from (4.6) under a similar interchange of shell and gravitational wave. This can be obtained by replacing (4.5) by the expression for Ψ_2 evaluated on $u = 0$. More simply, since in particular we are assuming the $l\ ,\ k\ ,\ s\ ,\ p \neq 0$, if (4.6) is multiplied by s and (4.1) is used we immediately arrive at

$$a\,s = b\,l\ . \tag{4.7}$$

We now have, on account of (3.4) and (4.2) with (4.6) holding, that when $v = 0\ ,\ u > 0$,

$$\phi_0 = b\ , \qquad \phi_2 = \frac{a}{1 + a^2 u^2}\ , \tag{4.8}$$

while conversely when $u = 0\ ,\ v > 0$,

$$\phi_2 = a\ , \qquad \phi_0 = \frac{b}{1 + b^2 v^2}\ . \tag{4.9}$$

We now make the assumption (2.20) from which we again arrive at (2.29) (on account of (4.8) and (4.9)). Introducing the barred coordinates $\bar{u}\ ,\ \bar{v}$ via $b\,v = \tan\bar{v}$ and $a\,u = \tan\bar{u}$ as before we have

$$\bar{\phi}_0 = \bar{\phi}_2\ . \tag{4.10}$$

It thus follows that when $\bar{v} = 0\ ,\ \bar{u} > 0$,

$$\bar{\phi}_0 = \bar{\phi}_2 = 1\ , \tag{4.11}$$

and similarly when $\bar{u} = 0\ ,\ \bar{v} > 0$. Now the initial data for the wave equation (2.22) reads

$$P(\bar{u}) = \log\left[\frac{a + (k - l)\ \tan\bar{u}}{a - (k + l)\ \tan\bar{u}}\right]\ , \tag{4.12}$$

and (written in a convenient form for performing the integration in (2.23))

$$Q(\bar{u}) = \frac{a\ \tan\bar{u} + (k + l)}{a - (k + l)\ \tan\bar{u}} + \frac{a\ \tan\bar{u} - (k - l)}{a + (k - l)\ \tan\bar{u}}\ . \tag{4.13}$$

The d'Alembert formula (2.23) gives the solution V in the barred coordinates as

$$V = \log\left[\frac{a\ \cos(\bar{u} - \bar{v}) + (k - l)\ \sin(\bar{u} - \bar{v})}{a\ \cos(\bar{u} + \bar{v}) - (k + l)\ \sin(\bar{u} + \bar{v})}\right]\ . \tag{4.14}$$

At this point it is useful to define the functions

$$F_1(\bar{u} - \bar{v}) \ = \ \cos(\bar{u} - \bar{v}) + \frac{(k - l)}{a}\ \sin(\bar{u} - \bar{v})\ , \tag{4.15}$$

$$F_2(\bar{u} + \bar{v}) \ = \ \cos(\bar{u} + \bar{v}) - \frac{(k + l)}{a}\ \sin(\bar{u} + \bar{v})\ . \tag{4.16}$$

We see that these are wave functions and also solutions of the unit frequency harmonic oscillator equation. In terms of them we can write U given by (4.4) and V by (4.14) in the simple form

$$e^{-U} = F_1\,F_2\ , \qquad e^{V} = \frac{F_1}{F_2}\ . \tag{4.17}$$

From this we see that

$$U_{\bar v}=V_{\bar u}\ ,\qquad U_{\bar u}=V_{\bar v}\ , \tag{4.18}$$

and using these in Maxwell's equations (2.2) and (2.3) written in the barred coordinates, together with (4.10) and the initial data (4.11), we immediately see that for $\bar u>0\ ,\ \bar v>0$ we have

$$\bar\phi_0=\bar\phi_2=1\ . \tag{4.19}$$

Equations (4.17) and (4.19) are very easy to calculate with. The Einstein–Maxwell field equations (2.5) and (2.6) in the barred coordinates easily reduce to

$$\bar M_{\bar u}=0=\bar M_{\bar v}\ , \tag{4.20}$$

and now it follows that the remaining Einstein–Maxwell field equations (2.7) and (2.8) are automatically satisfied. With $\bar M$ given in terms of M by (2.19) the initial data for $\bar M$ read: when $\bar v=0\ ,\ \bar u>0$, $\bar M=\log(a\,b)$ and when $\bar u=0\ ,\ \bar v>0$, $\bar M=\log(a\,b)$. Thus on account of (4.20) we have for $\bar u>0\ ,\ \bar v>0$,

$$\bar M=\log(a\,b)\ . \tag{4.21}$$

Finally it is straightforward to see from (4.17) that

$$V_{\bar u\bar u}-U_{\bar u}\,V_{\bar u}=0=V_{\bar v\bar v}-U_{\bar v}\,V_{\bar v}\ , \tag{4.22}$$

which together with (4.18) help to confirm that the Weyl tensor components (2.10)–(2.14) all vanish in the barred coordinates for $\bar u>0\ ,\ \bar v>0$ and so the collision space–time that we have constructed is indeed conformally flat.

It is interesting to restore the unbarred coordinates (u,v) using $b\,v=\tan\bar v$ and $a\,u=\tan\bar u$. The function U is given in the coordinates (u,v) by (4.4) which, in the light of (4.17), can be simplified to read

$$e^{-U}=\frac{[1+a\,b\,u\,v+(k-l)\,u-(p-s)\,v]\,[1-a\,b\,u\,v-(k+l)\,u-(p+s)\,v]}{(1+a^2u^2)\,(1+b^2v^2)}\ . \tag{4.23}$$

By (4.13) the function V takes the form

$$V=\log\left[\frac{1+a\,b\,u\,v+(k-l)\,u-(p-s)\,v}{1-a\,b\,u\,v-(k+l)\,u-(p+s)\,v}\right]\ . \tag{4.24}$$

We have made use of (4.6) and (4.7) to simplify this expression. Finally the functions $\phi_0\,,\ \phi_2\,,\ M$ are given by

$$\phi_0=\frac{b}{1+b^2v^2}\ ,\ \phi_2=\frac{a}{1+a^2u^2}\ ,\ M=\log(1+a^2u^2)\,(1+b^2v^2)\ . \tag{4.25}$$

The Bell–Szekeres solution is an allowable special case satisfying the conditions (4.1), (4.6) and (4.7) with $k = l = p = s = 0$ and we now see that (4.23)), (4.24) and (4.25) agree with (2.26), (2.31), (2.35) and (2.39) in this case.

5 Consequences of Conformal Flatness

We begin by examining the mathematical consistency with the field equations of the assumption of conformal flatness in the region of the space–time after the collision. From (2.10)–(2.14) we have the necessary and sufficient conditions for conformal flatness:

$$\begin{aligned} V_{vv} &= U_v\,V_v - M_v\,V_v \ , && (5.1)\\ U_u\,U_v &= V_u\,V_v \ , && (5.2)\\ V_{uu} &= U_u\,V_u - M_u\,V_u \ . && (5.3) \end{aligned}$$

Differentiating (5.2) with respect to v and using the field equations (given in (2.2)–(2.8)) we find that provided $\phi_0 \neq 0$ we must have

$$U_u\,\phi_0 = V_v\,\phi_2 \ , \tag{5.4}$$

and thus (2.3) implies

$$\phi_0 = \phi_0(v) \ . \tag{5.5}$$

Similarly differentiating (5.2) with respect to u we find that if $\phi_2 \neq 0$ then

$$U_v\,\phi_2 = V_u\,\phi_0 \ , \tag{5.6}$$

and so the Maxwell equation (2.2) implies

$$\phi_2 = \phi_2(u) \ . \tag{5.7}$$

From (5.5) and (5.7) we see in particular that we have the separation of variables (2.20) as a consequence of conformal flatness. Next differentiating (5.1) with respect to u, using the field equations, and assuming that $\phi_2 \neq 0$ we arrive at

$$\frac{d\phi_0}{dv} = -M_v\,\phi_0 \ , \tag{5.8}$$

while differentiating (5.2) with respect to v we find that if $\phi_0 \neq 0$ then

$$\frac{d\phi_2}{du} = -M_u\,\phi_2 \ . \tag{5.9}$$

On account of the field equation (2.8) we see from (5.2) that

$$M = A(u) + B(v) , \tag{5.10}$$

where the functions A and B are arbitrary. Using this in (5.8) and (5.9) we obtain

$$\phi_0 = c_0\, e^{-B(v)} , \qquad \phi_2 = c_2\, e^{-A(u)} , \tag{5.11}$$

where c_0 and c_2 are constants. We solve (5.4) and (5.6) for V_v and V_u to obtain

$$V_v = \frac{\phi_0}{\phi_2}\, U_u , \qquad V_u = \frac{\phi_2}{\phi_0}\, U_v . \tag{5.12}$$

It is now clear that with U given by (2.9) the equations (5.1)–(5.3) are satisfied. Using (5.8) and (5.9) the integrability conditions for (5.12) read

$$\phi_2^2(U_{vv} + U_v\, M_v) = \phi_0^2(U_{uu} + U_u\, M_u) . \tag{5.13}$$

Substituting from the field equations (2.5) and (2.6) this becomes

$$\phi_2^2(U_v^2 + V_v^2) = \phi_0^2(U_u^2 + V_u^2) , \tag{5.14}$$

and this equation is satisfied on account of (5.12).

To examine the status of (2.7) we use the first of (5.12) along with (5.9) to obtain

$$2\, V_{vu} = 2\, \frac{\phi_0}{\phi_2}\, (U_{uu} + U_u\, M_u) . \tag{5.15}$$

By (2.5) this reads

$$\begin{aligned} 2\, V_{vu} &= \frac{\phi_0}{\phi_2}\, (U_u^2 + V_u^2 + 4\, \phi_2^2) , \\ &= V_v\, U_u + V_u\, U_v + 4\, \phi_0\, \phi_2 , \end{aligned} \tag{5.16}$$

using (5.12) again. Thus we see that (2.7) is satisfied. From now on we therefore concentrate attention on (2.5) and (2.6).

In view of (2.19) we can take advantage of (5.11) to introduce barred coordinates $\bar{u}(u)$, $\bar{v}(v)$ via the differential equations

$$\frac{d\bar{u}}{du} = c_2\, e^{-A(u)} , \qquad \frac{d\bar{v}}{dv} = c_0\, e^{-B(v)} . \tag{5.17}$$

This has the immediate effect of having

$$\bar{\phi}_0 = \bar{\phi}_2 = 1 , \tag{5.18}$$

and, on account of (5.10), of also having

$$\bar{M} = \log(c_0\, c_2)\ . \tag{5.19}$$

The field equations (2.2)–(2.8) remain invariant in form under the transformation to the barred coordinates and hence (2.4) gives, in the barred system,

$$e^{-U} = \mathcal{F}(\bar{u}) + \mathcal{G}(\bar{v})\ , \tag{5.20}$$

where $\mathcal{F}\,,\ \mathcal{G}$ are arbitrary while the remaining field equations of interest, (2.5) and (2.6), simplify in the barred system to

$$2\,U_{\bar{u}\bar{u}} = U_{\bar{u}}^2 + V_{\bar{u}}^2 + 4\ , \tag{5.21}$$
$$2\,U_{\bar{v}\bar{v}} = U_{\bar{v}}^2 + V_{\bar{v}}^2 + 4\ . \tag{5.22}$$

We also have (5.12), which in the barred system reduces to

$$U_{\bar{u}} = V_{\bar{v}}\ , \qquad U_{\bar{v}} = V_{\bar{u}}\ . \tag{5.23}$$

Thus we will solve for the initial data functions $\mathcal{F}\,,\ \mathcal{G}$ using (5.21) and (5.22) in the form

$$2\,U_{\bar{u}\bar{u}} = U_{\bar{u}}^2 + U_{\bar{v}}^2 + 4 = 2\,U_{\bar{v}\bar{v}}\ . \tag{5.24}$$

The two equations we obtain for $\mathcal{F}\,,\ \mathcal{G}$ can be written in the form

$$\frac{d^2\mathcal{F}}{d\bar{u}^2} + 4\,\mathcal{F} = -\left(\frac{d^2\mathcal{G}}{d\bar{v}^2} + 4\,\mathcal{G}\right)\ , \tag{5.25}$$

and

$$(\mathcal{F} + \mathcal{G})\left(\frac{d^2\mathcal{G}}{d\bar{v}^2} - \frac{d^2\mathcal{F}}{d\bar{u}^2}\right) = \left(\frac{d\mathcal{G}}{d\bar{v}}\right)^2 - \left(\frac{d\mathcal{F}}{d\bar{u}}\right)^2\ . \tag{5.26}$$

The solutions of (5.25) are

$$\mathcal{F} = \alpha_0\ \sin 2\,\bar{u} + \beta_0\ \cos 2\,\bar{u} + k_0\ , \tag{5.27}$$
$$\mathcal{G} = \gamma_0\ \sin 2\,\bar{v} + \delta_0\ \cos 2\,\bar{v} - k_0\ , \tag{5.28}$$

where k_0 is a separation constant and $\alpha_0, \beta_0,\ \gamma_0, \delta_0$ are constants of integration. Substitution into (5.26) yields

$$\alpha_0^2 + \beta_0^2 = \gamma_0^2 + \delta_0^2\ . \tag{5.29}$$

Writing $\alpha_0 + i\beta_0 = R_0\, e^{2i\xi_0}$ and $\gamma_0 + i\delta_0 = R_0\, e^{2i\eta_0}$ we can write (5.20) as

$$e^{-U} = C\,\{\cos(\bar{u}{+}\bar{v}){+}\cot(\xi_0{+}\eta_0)\ \sin(\bar{u}{+}\bar{v})\}\,\{\cos(\bar{u}{-}\bar{v}){-}\tan(\xi_0{-}\eta_0)\ \sin(\bar{u}{-}\bar{v})\}\ , \tag{5.30}$$

with

$$C = 2\,R_0\,\sin(\xi_0+\eta_0)\,\cos(\xi_0-\eta_0) = \text{constant}\ . \tag{5.31}$$

Using this in (5.23) we solve for V to obtain

$$e^V = W_0\left(\frac{\cos(\bar u-\bar v)-\tan(\xi_0-\eta_0)\,\sin(\bar u-\bar v)}{\cos(\bar u+\bar v)+\cot(\xi_0+\eta_0)\,\sin(\bar u+\bar v)}\right)\ , \tag{5.32}$$

where W_0 is a constant of integration. For substitution into the line–element (2.1) we now have the functions

$$\begin{aligned} e^{-U+V} &= C_1\,\{\cos(\bar u-\bar v)-\tan(\xi_0-\eta_0)\,\sin(\bar u-\bar v)\}^2\ , && (5.33)\\ e^{-U-V} &= C_2\,\{\cos(\bar u+\bar v)+\cot(\xi_0+\eta_0)\,\sin(\bar u+\bar v)\}^2\ , && (5.34)\end{aligned}$$

where

$$\begin{aligned} C_1 &= 2\,W_0\,R_0\,\sin(\xi_0+\eta_0)\,\cos(\xi_0-\eta_0)\ , && (5.35)\\ C_2 &= 2\,\frac{R_0}{W_0}\,\sin(\xi_0+\eta_0)\,\cos(\xi_0-\eta_0)\ . && (5.36)\end{aligned}$$

These constants can be absorbed by a re-scaling of the coordinates x, y which is equivalent to putting, without loss of generality, $C = W_0 = 1$ in (5.30) and (5.32). Comparing now the expressions for $\bar M, \bar\phi_0,\ \bar\phi_2,\ U,\ V$ which we have derived here in (5.18), (5.19), (5.30) (with $C = 1$) and (5.32) (with $W_0 = 1$) with the corresponding expressions (4.19), (4.21) and (4.17), obtained as the solution of the collision problem in section 4, we see that they are identical provided the constants $c_0, c_2, \xi_0,\ \eta_0$ in this section are related to the constants a, b, k, l in section 4 by

$$c_0\,c_2 = a\,b\ ,\quad \cot(\xi_0+\eta_0) = -\frac{(k+l)}{a}\ ,\quad \tan(\xi_0-\eta_0) = -\frac{(k-l)}{a}\ . \tag{5.37}$$

We have thus established that if the space–time region after the collision of the light–like signals is a solution of the vacuum Einstein–Maxwell field equations and if, in addition, it is conformally flat, then the colliding light–like signals have to be those related combinations of shells, impulsive gravitational waves and electromagnetic shock waves described in section 4.

6 Discussion

We have focussed on the conformal flatness property of the Bell–Szekeres solution and carried that into the generalization derived in section 4. There is however a further property of the Bell–Szekeres solution which is inherited by the generalization. To see this we write the line–element of the solution in section 4 in the barred coordinates. It is given by

$$ds^2 = -F_1^2 dx^2 - F_2^2 dy^2 + 2\,(a\,b)^{-1} d\bar{u}\,d\bar{v}\ , \tag{6.1}$$

with F_1 and F_2 given by (4.15) and (4.16). Introducing coordinates $\xi = \bar{u} - \bar{v}$ and $\eta = \bar{u} + \bar{v}$ we have

$$ds^2 = ds_1^2 + ds_2^2\ , \tag{6.2}$$

with

$$ds_1^2 = -F_1^2(\xi)\,dx^2 - \frac{1}{2\,a\,b}\,d\xi^2\ , \tag{6.3}$$

$$ds_2^2 = -F_2^2(\eta)\,dy^2 + \frac{1}{2\,a\,b}\,d\eta^2\ . \tag{6.4}$$

Hence we see that the collision space–time has decomposed into the Cartesian product of a pair of two–dimensional space–times having line–elements (6.3) and (6.4). The Gaussian curvature of the 2–space with line–element (6.3) is $K_1 = \mp 2\,a\,b$ while the Gaussian curvature of the 2–space with line element (6.4) is $K_2 = \pm 2\,a\,b$ - in each case we have here a 2–space of constant Gaussian curvature of opposite sign. This is a property that the collision space–times derived in section 4 share with the Bell–Szekeres space–time. It is well–known that the Bell–Szekeres space–time coincides with the Bertotti–Robinson [2][3] space–time which was originally identified as the four dimensional Einstein–Maxwell vacuum space–time having this topological property. The solution given in section 4 represents only *a portion* of the Bertotti–Robinson space–time because, depending upon the sign of the product $a\,b$, either the x coordinate is periodic with period $2\,\pi\,a/\sqrt{a^2 + (k-l)^2} < 2\,\pi$ or the y coordinate is periodic with period $2\,\pi\,a/\sqrt{a^2 + (k+l)^2} < 2\,\pi$. In both cases the period of the coordinate is $2\,\pi$ in the Bell–Szekeres special case.

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A In-Coming Signal

The line–element of the space–time containing the history of the in–coming signal can be written in the form

$$\begin{aligned} ds^2 &= -\left(\cos a\,u'_+ + \frac{(k-l)}{a}\,\sin a\,u'_+\right)^2 dx^2 - \left(\cos a\,u'_+ - \frac{(k+l)}{a}\,\sin a\,u'_+\right)^2 dy^2 \\ &\quad +2\,du'\,dv\ . \end{aligned} \tag{A.1}$$

Here $u'_+ = u'\,\vartheta(u')$ where $\vartheta(u')$ is the Heaviside step function which is equal to zero if $u' < 0$ and equal to unity if $u' > 0$. Direct calculation of the Ricci tensor on the half null tetrad given via the 1–forms

$$\theta^1 = \left(\cos a\,u'_+ + \frac{(k-l)}{a}\,\sin a\,u'_+\right) dx = -\theta_1\ , \tag{A.2}$$

$$\theta^2 = \left(\cos a\,u'_+ - \frac{(k+l)}{a}\,\sin a\,u'_+\right) dy = -\theta_2\ , \tag{A.3}$$

$$\theta^3 = dv = \theta_4\ , \tag{A.4}$$

$$\theta^4 = du' = \theta_3\ , \tag{A.5}$$

results in $R_{ab} \equiv 0$ except for

$$R_{44} = -2\,l\,\delta(u') - 2\,a^2\vartheta(u')\ , \tag{A.6}$$

where $\delta(u')$ is the Dirac delta function. The a^2–term in (A.6) is due to a vacuum electromagnetic field. Taking the potential 1–form to be

$$A = \left(\sin a\,u'_+ - \frac{(k-l)}{a}\,\cos a\,u'_+\right)\,dx\ , \tag{A.7}$$

we obtain the field F and its dual ${}^*F$ given respectively by

$$F = -a\,\vartheta(u')\,\theta^2 \wedge \theta^4\ , \qquad \text{and} \qquad {}^*F = -a\,\vartheta(u')\,\theta^1 \wedge \theta^4\ . \tag{A.8}$$

It follows trivially that Maxwell's vacuum field equations are satisfied by F. Calculation of the electromagnetic energy–momentum tensor components E_{ab} on the tetrad reveals that all components vanish except

$$E_{44} = -a^2\,\vartheta(u')\ . \tag{A.9}$$

Thus (A.6) may be re–written in the form

$$R_{44} - 2\,E_{44} = -2\,l\,\delta(u')\ . \tag{A.10}$$

Thus the Einstein–Maxwell field equations with a light–like shell source (provided $l \neq 0$) are satisfied by the metric tensor given via the line–element (A.1). The only non–vanishing Weyl tensor component in Newman–Penrose notation is

$$\Psi_4 = -R_{1414} + \frac{1}{2}\,R_{44} = -k\,\delta(u')\ , \tag{A.11}$$

indicating that the signal is accompanied by an impulsive gravitational wave provided $k \neq 0$. This Weyl tensor is type N in the Petrov classification with $\partial/\partial v$ as degenerate principal null direction. Hence the key equations for physically interpreting the signal are (A.9)–(A.11). When $u' > 0$ the transformation $a\,u = \tan a\,u'$ applied to (A.1) yields the initial data quoted in (3.1)–(3.4).